A Kinetic Framework to Study Opinion Dynamics in Multi-Agent Systems

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Abstract. In this paper we summarize an analytic framework to study opinion dynamics. Such a framework is inspired from physics and, in particular, from the kinetic theory of gases with different types of molecules, known as gas mixtures. By reinterpreting the molecules of gases as agents, and the collisions among molecules as interactions among agents, we show how to adapt the kinetic approach to the study of opinion dynamics in multi-agent systems. First, we briefly introduce the key ingredients of the kinetic theory of gas mixtures to clarify nomenclature and to emphasize the substantial differences with the proposed framework. Then, we show how the kinetic approach can be used to study opinion dynamics in multi-agent systems, and how a variety of sociological phenomena can be accommodated in the proposed framework. In the last part of the paper we focus on two sociological phenomena, namely compromise and diffusion, and we describe analytic results concerning stationary profiles of the distribution of opinion, and of the temporal evolution of the average opinion and of the variance of the opinion.

1 Introduction

Besides a notable interest witnessed in the research on agent-based models (see, e.g., [5]), a significant interest on opinion dynamics is found in many research areas. Opinion dynamics and the effects of social interactions are studied in the context of sociology to derive scientific theories and models to explain collective behaviors and socio-cultural differentiation (see, e.g., [8]). Opinion dynamics has been investigated in the field of artificial life [7], which proposes common models for many types of agents, such as: plants and animals in ecosystems, vehicles moving in the traffic, pedestrians walking indoor or outdoor, and actors in large-scale organizations. Finally, opinion dynamics has been studied in various fields related to artificial intelligence, such as cooperative multi-agent systems (see, e.g., [2, 17]). The interest in the study of opinion dynamics is still very high because of its central role in very challenging applications like social networks.

The majority of existing agent-based models used to study opinion dynamics are based on simulations, and obtained results are valid within the scope of the type of systems that have been actually simulated. On the contrary, in this paper we propose to study opinion dynamics from an analytic point of view, whose results are valid within the scope of the hypotheses used to obtain them. The analytic approach that we adopt is inspired by physics, and we acknowledge that other models of opinion dynamics inspired by physics are documented in the literature, such as those based on statistical mechanics [21] or on thermodynamics and Brownian motion [20]. In detail, the approach that we consider in this paper is based on the kinetic theory of gas mixtures, a branch of physics that aims at formalizing observable macroscopic properties of gas mixtures, which are composed of different types of molecules. The observable macroscopic properties of interest in kinetic theory are, e.g., the pressure and the temperature of the gas mixture, and the theory moves from a description of microscopic interactions among molecules.

Kinetic theory studies the microscopic collisions among molecules in gases and it relies on the details of such collisions to derive macroscopic properties of gases, using proper balance equations. Similarly, when dealing with opinion dynamics, one typically starts from the description of microscopic interactions among agents and aims at deriving macroscopic properties of the system concerning, e.g., the temporal evolution of the average opinion [18,23]. Hence, by reinterpreting the molecules of gases as agents it is possible to adapt the kinetic framework to analyse opinion dynamics in multi-agent systems. Notably, very few results of kinetic theory of gas mixtures are applicable to the study of opinion dynamics because microscopic collisions among the molecules are significantly different from microscopic interactions among agents, as considered typically by sociologists. For this reason, while the general approach of kinetic theory remains valid, the two theories are substantially different. In this paper, we show that it is possible to derive analytic relations between the microscopic laws that describe the effects of interactions among agents and the distribution of the opinion after a sufficiently large number of interactions, which is normally called *stationary profile*. Moreover, we show that the use of the proposed kinetic framework allows macroscopic characteristics of the system, such as the average opinion or the variance of the opinion, to be determined analytically.

This paper is organized as follows. Section 2 outlines the basic nomenclature of the kinetic theory of gas mixtures. Section 3 presents the proposed framework using the nomenclature of kinetic theory. Section 4 discusses relevant sociological phenomena which can be accommodated in the proposed framework, and it shows analytic results obtained considering some interesting sociological phenomena. Finally, Section 5 concludes the paper and outlines ongoing and future work.

2 Overview of Kinetic Theory of Gas Mixtures

Classic kinetic theory of gas mixtures formalizes an explanation of observable macroscopic properties of gas mixtures, e.g., pressure and temperature, starting from the analysis of microscopic collisions among molecules. The basic idea of the theory is to describe microscopic collisions among molecules of a gas from a probabilistic point of view, with the objective of deriving a description of observ-

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able macroscopic properties by means of proper balance equations. Normally, kinetic theory of gas mixtures is described by first studying the case of gases with only one type of molecules. In this particular case, the theory assumes the existence of a single *distribution function* $f(\underline{x}, \underline{v}, t)$, which represents the density of molecules at position $\underline{x} \in \mathbb{R}^3$ with velocity $\underline{v} \in \mathbb{R}^3$ at time $t \geq 0$. Typically, it is assumed that the distribution function evolves according to the Boltzmann equation, which is an integro-differential equation whose unknown is function $f(\underline{x}, \underline{v}, t)$. The spatially homogeneous Boltzmann equation is

$$\frac{\partial f}{\partial t} = \mathcal{Q}(f, f)(\underline{x}, \underline{v}, t) \tag{1}$$

where the term on the right hand side is related to the effects of binary collisions among molecules, and for this reason, $\mathcal{Q}(f, f)$ is called *collisional operator*. Given the distribution function $f(\underline{x}, \underline{v}, t)$, it is possible to define macroscopic properties typically associated with gases. First, by integrating $f(\underline{x}, \underline{v}, t)$ with respect to $\underline{v}$, one obtains

$$n(\underline{x}, t) = \int_{\mathbb{R}^3} f(\underline{x}, \underline{v}, t)\, \mathrm{d}_3\underline{v} \tag{2}$$

which represents the number of molecules per unit volume at time t with position $\underline{x}$. The average velocity of molecules, instead, can be computed as

$$\underline{u}(\underline{x}, t) = \frac{1}{n(\underline{x}, t)} \int_{\mathbb{R}^3} \underline{v}\, f(\underline{x}, \underline{v}, t)\, \mathrm{d}_3\underline{v}. \tag{3}$$

Finally, the local temperature $T(\underline{x}, t)$ at time t can be computed according to the following equality

$$\frac{3}{2} n(\underline{x}, t) k T(\underline{x}, t) = m \int_{\mathbb{R}^3} |(\underline{v} - \underline{u}(\underline{x}, t))|^2 f(\underline{v}, \underline{x}, t) \mathrm{d}_3\underline{v} \tag{4}$$

where k is the Boltzmann constant, and m is the mass of molecules.

Observe that the Boltzmann equation (1) is relative to a gas where all the molecules are equal. Instead, a gas is typically composed of different kinds of molecules. For this reason, kinetic theory of gas mixtures, namely of gases with different types of molecules, typically called *species*, was introduced. In this case, denoting as $M \geq 1$ the number of species in the considered gas, M distribution functions $\{f_s\}_{s=1}^M$ are defined. When considering a gas mixture, the temporal evolution of each density function can be described as

$$\frac{\partial f_s}{\partial t} = \sum_{r=1}^{M} \mathcal{Q}_{sr}(f_s, f_r)(\underline{x}, \underline{v}, t) \qquad 1 \leq s \leq M \tag{5}$$

where the right-hand side represents the collisional operator relative to species s, which is written as the sum of the collisional operators $\mathcal{Q}_{sr}(f_s, f_r)$ relative to collisions among molecules of species s and any species $1 \leq r \leq M$. Observe that the collisional operator relative to species s depends on the distribution functions $\{f_s\}_{s=1}^M$ of all species. As in the case with only one species, it is possible to compute macroscopic properties, as follows. The number of molecules of species s per unit volume at time t with position $\underline{x}$ is obtained by integrating the distribution function relative to species s with respect to $\underline{v}$

$$n_s(\underline{x}, t) = \int_{\mathbb{R}^3} f_s(\underline{x}, \underline{v}, t)\, \mathrm{d}_3\underline{v} \qquad 1 \leq s \leq M. \tag{6}$$

The average velocity of molecules of species s can be computed as

$$\underline{u}_s(\underline{x}, t) = \frac{1}{n_s(\underline{x}, t)} \int_{\mathbb{R}^3} \underline{v}\, f_s(\underline{x}, \underline{v}, t)\, \mathrm{d}_3\underline{v} \qquad 1 \leq s \leq M. \tag{7}$$

Finally, the local temperature $T_s(\underline{x}, t)$ of species s, for all $s \in \{1, \ldots, M\}$, is computed according to

$$\frac{3}{2} n_s(\underline{x}, t) k T_s(\underline{x}, t) = m_s \int_{\mathbb{R}^3} |(\underline{v} - \underline{u}_s(\underline{x}, t))|^2 f_s(\underline{v}, \underline{x}, t)\, \mathrm{d}_3\underline{v} \tag{8}$$

where m_s is the mass of molecules of species s.

3 A Kinetic Framework for Opinion Dynamics

In this section, we use the kinetic approach described in Section 2 to investigate opinion dynamics in multi-agent systems. Before going into the details of the proposed framework, it is worth mentioning the basic assumptions which underlie the framework. Opinion has been modeled in the literature both as a discrete [24] and as a continuous [6] variable. While discrete models are typically used to address situations where a finite number of options are available, e.g., in political elections, continuous models are typically used to study opinions related to a single topics, varying from strongly disagree to completely agree. In the proposed framework, we model opinion as a continuous variable. In general, the interactions that govern the dynamics of opinion may involve two [13] or more [10] agents, and in the first case they are called *binary interactions*. In the proposed framework, we focus on binary interactions. Finally, opinion models differ due to the phenomena used to describe how the opinions of agents change after single interactions. Opinion may change because of a variety of phenomena, which has been classified into an accepted nomenclature by sociology. Single models typically include a specific subset of such phenomena depending on the specific characteristics of the studied multi-agent systems, on targeted application scenarios, and on the background of authors. The proposed framework accommodates various phenomena to update the opinions of interacting agents, as detailed in Section 4.

As in kinetic theory, we aim at studying interesting macroscopic properties of a system made of interacting peers, now called agents and not molecules, starting from the analysis of the effects of single interactions. Inspired by kinetic theory, we assume that all agents can interact with each other, and we consider binary interactions among agents. While in kinetic theory the molecules of gases are associated with physical parameters, such as their positions and velocities, here we assume that each agent is associated with a single scalar parameter v which represents its opinion, and we assume that v varies continuously in a proper closed interval, denoted as $I \subseteq \mathbb{R}$.

We first start with a system in which all agents are equal, and then we extend the framework to a system made of different types of agents. When agents are all equal, we assume the existence of a distribution function for the opinion, denoted as $f(v, t)$, which represents the number of agents with opinion v at time t, and which is defined for each opinion $v \in I$ and for each time $t \geq 0$. Obviously, due to its definition, function $f(v, t)$ is non-negative. Using the distribution function $f(v, t)$, the number of agents at time t can be written as

$$n(t) = \int_I f(v, t) \mathrm{d}v, \tag{9}$$

and the average opinion $u(t)$ at time t can be computed as

$$u(t) = \frac{1}{n(t)} \int_I v f(v, t) \mathrm{d}v. \tag{10}$$

The number of agents at time t is obviously related to the number of molecules in a gas as in (2), while the average opinion $u(t)$ is related to the average velocity of the molecules of a gases as in (3). Given the average opinion $u(t)$, it is also possible to compute the variance of the opinion as

$$\sigma_f^2(t) = \frac{1}{n(t)} \int_I (v-u)^2 f(v,t) \mathrm{d}v. \tag{11}$$

Observe that $\sigma_f^2(t)$ is related to the temperature of gases in (4).

As in kinetic theory, we assume that function $f(v,t)$ evolves on the basis of the Boltzmann equation (1), whose form is now

$$\frac{\partial f}{\partial t} = \mathcal{Q}(f,f)(v,t). \tag{12}$$

Using the same nomenclature of kinetic theory, we refer to $\mathcal{Q}$ as collisional operator. However, we remark that the explicit expression of the collisional operator used in kinetic theory is substantially different from that adopted to study opinion dynamics, as outlined in Section 4.

Following consolidated practice in kinetic theory, we focus on the *weak form* of the Boltzmann equation, as usual when trying to find solutions of a non-trivial integro-differential equation. The weak form of a differential equation is obtained by multiplying both sides of the considered equation by a smooth function with compact support, typically called *test function*, and then integrating the obtained equation. Hence, the weak form of the Boltzmann equation (12) is

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_I f(v,t)\phi(v) \mathrm{d}v = \int_I \mathcal{Q}(f,f)(v,t)\phi(v) \mathrm{d}v \tag{13}$$

where the right-hand side is called *weak form of the collisional operator* $\mathcal{Q}$ with respect to test function $\phi(v)$. Functions that satisfy (13) are called *weak solutions* of the Boltzmann equation, and they are useful to study macroscopic properties of a system, as follows:

1. The left-hand side of (13) with $\phi(v) = 1$ represents the derivative, with respect to time, of the number of agents $n(t)$;
2. The left-hand side of (13) with $\phi(v) = v$ is proportional to the derivative, with respect to time, of the average opinion $u(t)$; and
3. The left-hand side of (13) with $\phi(v) = (v-u)^2$ is proportional to the derivative, with respect to time, of the variance of the opinion $\sigma_f^2(t)$.

The explicit expression of the weak form of the collisional operator which appears at the right-hand side of (13) is essential to analyse the temporal evolution of the characteristics of the systems. Moreover, the weak form of the Boltzmann equation is also used to derive the stationary profiles of the opinion [22].

The proposed kinetic framework can be also used to study multi-agent systems where agents have different characteristics. Assuming that the agents of the considered multi-agent systems are divided into $M \geq 1$ *classes*, each of which has different characteristics, it is possible to define M functions $\{f_s(v,t)\}_{s=1}^M$. The number of agents of class s at time t can be written as

$$n_s(t) = \int_I f_s(v,t) \mathrm{d}v \qquad 1 \leq s \leq M, \tag{14}$$

and the total number of agents is

$$n(t) = \sum_{s=1}^M n_s(t). \tag{15}$$

The average opinion of agents of class s can be computed as

$$u_s(t) = \frac{1}{n_s(t)} \int_I f_s(v,t) v \mathrm{d}v \qquad 1 \leq s \leq M, \tag{16}$$

and it is the analogous of the average velocity of each species in a gas mixture in (7). Given the average opinions of all classes, it is possible to compute the global average opinion of the system at time t

$$u(t) = \frac{1}{n(t)} \sum_{s=1}^M n_s(t) u_s(t). \tag{17}$$

Finally, the standard deviation of the opinion relative to agents of class s can be computed as

$$\sigma_s^2(t) = \frac{1}{n_s(t)} \int_I (v-u_s)^2 f_s(v,t) \mathrm{d}v. \tag{18}$$

Observe that $\sigma_s^2(t)$ is related to the temperature of each species in a gas mixture (8). As in kinetic theory, we assume that each distribution function $f_s(v,t)$ evolves according to the Boltzmann equation

$$\frac{\partial f_s}{\partial t}(v,t) = \sum_{r=1}^M \mathcal{Q}_{sr}(f_s, f_r) \qquad 1 \leq s \leq M. \tag{19}$$

From (19) it is evident that the collisional operator relative to class s corresponds to the sum of the contributions $\mathcal{Q}_{sr}(f_s, f_r)$ of the collisions between the agents of class s with the agents of class $r \in \{1, \dots, M\}$. As in the case where agents are all equal, the study of the weak form of the Boltzmann equation, which can be written as

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_I f_s(v,t)\phi(v) \mathrm{d}v = \sum_{r=1}^M \int_I \mathcal{Q}_{sr}(f_s, f_r)\phi(v) \mathrm{d}v, \tag{20}$$

allows deriving differential equations relative to:

1. The number of agents of each class, using test function $\phi(v) = 1$;
2. The average opinion of each class, using test function $\phi(v) = v$; and
3. The variance of the opinion of each class, corresponding to the test function $\phi(v) = (v-u_s)^2$.

4 The Collisional Operator for Opinion Dynamics

The explicit expression of the collisional operator in the Boltzmann equation strongly depends on the details of the description of the effects of single interactions among agents. Hence, proper microscopic interaction rules need to be defined to be able to derive the explicit expression of the Boltzmann equation and, hence, to analytically study opinion dynamics.

4.1 Supported Sociological Phenomena

The following is a non-exhaustive list of the most common sociological phenomena which can be used to model how opinions change after interactions:

- *Compromise.* The tendency of agents to change their opinions towards those of agents they interact with [1].
- *Diffusion.* The process according to which the opinion of each agent can be influenced by the social context [4].
- *Homophily.* The tendency of individuals to communicate only with those with similar opinions [16].
- *Negative Influence.* The phenomena according to which agents evaluate their peers on the basis of some parameters and they only interact with those with positive scores [9].

- *Opinion Noise.* The phenomena according to which a random additive quantity leads to arbitrary opinion changes with small probability [19].
- *Striving for Uniqueness.* The process that is based in the idea that agents may want to distinguish from peers and, therefore, they may decide to change their opinions if too many peers share the same opinion [10].

All mentioned phenomena can be accommodated in the proposed framework by means of specific terms in the expression of the collisional operator, as detailed in various papers [3, 11–15]. Among the wide variety of such phenomena, in this paper we focus on compromise, which is the key ingredient of almost all models of opinion dynamics models, and on diffusion, which allows modeling systems where consensus is not reached.

4.2 Examples of Compromise and Diffusion

For the sake of simplicity, the following discussions assume that the multi-agent system contains only one class of agents, i.e., that all agents are equal. Similar, but more involved, results can be obtained considering different classes of agents (see, e.g., [3]). The simplest rule for changing the opinions of two interacting agents involve only compromise, namely the sociological phenomenon for which the opinions of two interacting agents get closer. Mathematically, if only compromise is considered, the effects of an interaction among two agents can be modeled as

$$\begin{cases} v' = v + \gamma(w - v) \\ w' = w + \gamma(v - w) \end{cases} \tag{21}$$

where v' and w' are the post-interaction opinions of the two agents and γ is a parameter in $(0, 1)$ which quantifies compromise. Observe that negative values of γ may be used, instead, to quantify the phenomenon of negative influence. As shown in [14], rules (21) can be used to derive the explicit expression of the weak form on the collisional operator as

$$\int_I \mathcal{Q}(f, f)\phi(v)\mathrm{d}v = \beta \int_{I^2} f(v)f(w) \left(\phi(v') - \phi(v)\right) \mathrm{d}v\mathrm{d}w \tag{22}$$

where β represents the probability of interaction between two agents. It is easy to observe that if we set $\phi(v) = 1$ in (22), the weak form of the collisional operator is 0 and, hence, the weak form of the Boltzmann equation is

$$\frac{\mathrm{d}}{\mathrm{d}t} n(t) = 0, \tag{23}$$

which means that the number of agents in the system is constant, and called n. If we consider, instead, test function $\phi(v) = v$, simple algebraic manipulations of (22) show that the weak form of the collisional operator equals 0. Then, the weak form of the Boltzmann equation relative to $\phi(v) = v$ is

$$\frac{\mathrm{d}}{\mathrm{d}t} u(t) = 0 \tag{24}$$

so it can be concluded that the average opinion is constant and called u. Finally, considering test function $\phi(v) = (v - u)^2$, the weak form of the Boltzmann equation can be written as the following first-order differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t} \sigma_f^2(t) = -2\beta n\gamma(1 - \gamma)\sigma_f^2(t) \tag{25}$$

from which, recalling that $\gamma \in (0, 1)$, it can be easily obtained that the variance of the opinion $\sigma_f^2(t)$ exponentially tends to 0 as t increases. This shows analytically that the opinions of all agents converge to the same value, which, according to (24), corresponds to the initial average opinion. Hence, consensus is reached, in agreement with other models involving only compromise [1].

Instead of considering only compromise, let us introduce also diffusion, which is modeled as a stochastic addend in the rules to update the opinions of two interacting agents. More precisely, we consider the following rules

$$\begin{cases} v' = v + \gamma(w - v) + \eta D(|v|) \\ w' = w + \gamma(v - w) + \eta_* D(|w|) \end{cases} \tag{26}$$

where η and η_* are two random variables with the same distribution $\vartheta(\cdot)$ whose support is denoted as B, and $D(\cdot)$ is the so called *diffusion function*. In this case, the explicit expression of the weak form of the collisional operator is

$$\begin{aligned} \mathcal{Q}(f, f) = \int_{I^2} \int_{B^2} \vartheta(\eta)\vartheta(\eta_*) f(v)f(w) \cdot \\ \left(\phi(v') - \phi(v)\right) \mathrm{d}v\mathrm{d}w\mathrm{d}\eta\mathrm{d}\eta_*. \end{aligned} \tag{27}$$

It is possible to show that the number of agents and the average opinion are conserved, as in the case without diffusion [12, 13]. Concerning the variance of the opinion, simple algebraic manipulation of the weak form of the Boltzmann equation relative to $\phi(v) = (v - u)^2$ results in the following equality

$$\frac{\mathrm{d}}{\mathrm{d}t} \sigma_f^2(t) = -2n\gamma(1 - \gamma)\sigma_f^2(t) + \sigma^2 \int_I f(v)D^2(|v|)\mathrm{d}v. \tag{28}$$

Due to the second addend at the right-hand side, in this case we cannot conclude that the variance of the opinion tends to 0 as the number of interactions increases. As a matter of fact, due to the contribution of diffusion, consensus is not reached in this case and, hence, opinions do not converges to a single value. In order to simplify notation, let us introduce the function

$$g(v, \tau) = f(v, t) \tag{29}$$

where τ is a temporal variable proportional to t according to parameter γ which appears in (26), namely

$$\tau = \gamma t. \tag{30}$$

The stationary distribution of the opinion, also known as stationary profile, can be found analytically by solving the following second order differential equation [22]

$$\frac{\lambda}{2} \frac{\partial^2}{\partial v^2}(D(|v|)^2 g) + \frac{\partial}{\partial v}((v - u)g) = 0, \tag{31}$$

with

$$\lambda = \sigma^2/\gamma. \tag{32}$$

and σ is the standard deviation of the random variables η and η_*. Observe that λ is the ratio of σ^2, which is related to diffusion, and γ, which is related to compromise. Hence, values of λ close to 0 correspond to models where compromise has a stronger impact than diffusion ($\gamma >> \sigma^2$). At the opposite, if large values of λ are considered, the contribution of diffusion is more important than that of compromise ($\sigma^2 >> \gamma$). In [13] it is shown that setting

$$D(|v|) = 1 - |v|$$

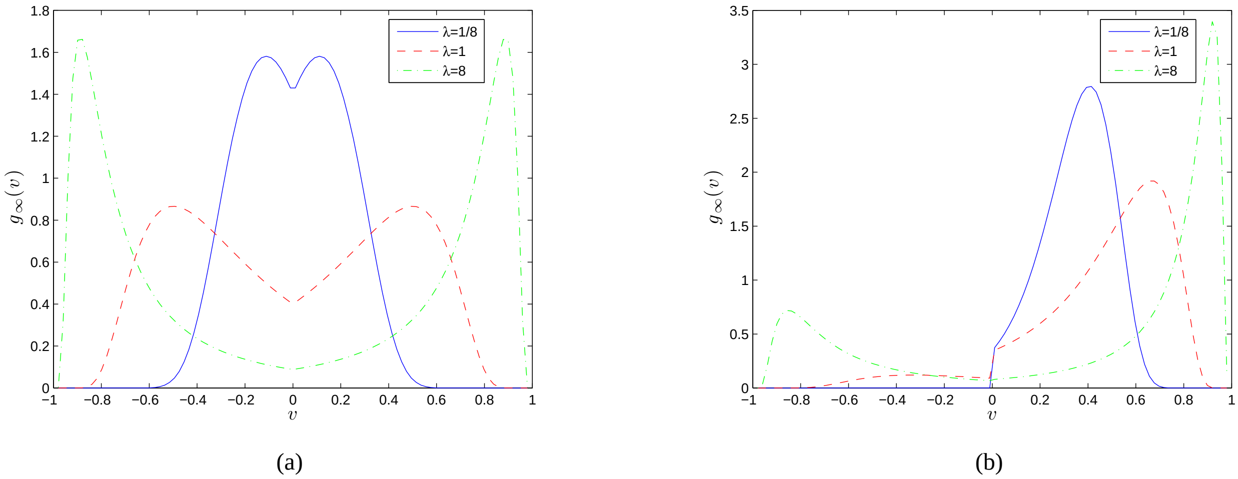


Figure 1. The stationary profiles in (33) are shown for $\lambda = 1/8$ (blue line), $\lambda = 1$ (red line), and $\lambda = 8$ (green line) and for: (a) $u = 0$; (b) $u = 1/3$.

in (26) leads to the following stationary profile

$$g_\infty(v) = c_{u,\lambda}(1 - |v|)^{-2-\frac{2}{\lambda}} \exp\left(-\frac{2(1-uv)}{\lambda(1-|v|)}\right) \tag{33}$$

wheret $c_{u,\lambda}$ is a constant that must be set to satisfy

$$\int_I g_\infty(v) = n. \tag{34}$$

The properties of the stationary profile in (33) differ on the basis of the values of the parameters involved. In particular, $g_\infty(v)$ can have one or two maxima, depending on the values of u and λ, as shown in [13]. In Fig. 1, some illustrative results are shown for different values of λ, namely: $\lambda = 1/8$ (blue line); $\lambda = 1$ (red line); and $\lambda = 8$ (green line). Fig. 1 (a) refers to the case in which the average opinion is $u = 0$. In this case, the stationary profiles are symmetric functions and, regardless of the value of λ, they have two points of maximum. Fig. 1 (b), instead, refers to the case in which the average opinion is $u = 1/3$, and in this case, the stationary profiles are not symmetric. If $\lambda = 1/8$, the stationary profile has only one point of maximum, which is positive. In the remaining cases, namely with $\lambda = 1$ and $\lambda = 8$ the stationary profiles have two points of maximum and the value of the positive maximum is far larger than that of the negative one. Instead, if we consider the diffusion function

$$D(|v|) = 1 - v^2$$

in (26), it is possible to show that the stationary profile has the following explicit expression [12]

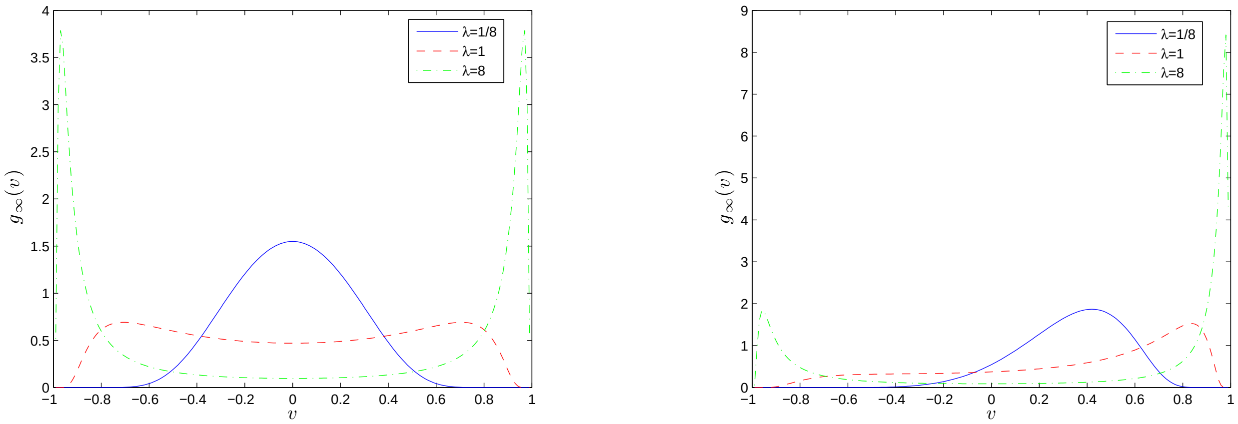
$$\begin{aligned} g_\infty(v) = & c_{u,\lambda}(1+v)^{-2+\frac{u}{2\lambda}}(1-v)^{-2-\frac{u}{2\lambda}} \cdot \\ & \exp\left(\frac{uv-1}{\lambda(1-v^2)}\right) \end{aligned} \tag{35}$$

where, once again, $c_{u,\lambda}$ must be set to satisfy (34). In this case, the stationary profile may have one, two, or three maxima, depending on the values of u and λ, as shown in [12]. In Fig. 2, some illustrative results are shown for different values of λ, namely: $\lambda = 1/8$ (blue line); $\lambda = 1$ (red line); and $\lambda = 8$ (green line). Fig. 2 (a) refers to the case in which the average opinion is $u = 0$. In this case, the stationary profiles are symmetric functions. If $\lambda = 1/8$, the stationary profile has one point of maximum, corresponding to $v = 0$, while the stationary profiles relative to $\lambda = 1$ and $\lambda = 8$ have two points of maximum. Fig. 2 (b) refers to the case in which the average opinion is $u = 1/3$. In this case, the stationary profiles are not symmetric. The stationary profiles corresponding to $\lambda = 1/8$ and $\lambda = 1$ have two points of maximum, while if $\lambda = 8$ the stationary profile has one point of maximum, which is positive.

As stated in the Section 1, when dealing with problems related to opinion dynamics, one may be interested in deriving the opinion distribution that is obtained as a consequence of given interaction rules which describe the effects of interactions among agents. This can be done also by simulations, even though the results may be less significant than those obtained analytically, as they depend on various factors like the duration of the simulations, the parameters used for the simulations, and the numerical stability of the simulation process. At the opposite, one may be interested in finding interaction rules leading to a desired distribution of opinion, which can correspond to collective agreement or to other profiles. Analytic approaches may be useful to address also these problems because the proper setting of the parameters involved in the model can be used as a method to obtain the desired stationary profiles.

5 Conclusions and Future Work

In this paper, we discussed a class of models to study the dynamics of opinion in multi-agent systems using an analytic framework inspired by kinetic theory of gas mixtures, a branch of physics which studies the macroscopic properties of gases starting from the analysis of microscopic collisions among molecules. We remark that this work is only inspired by kinetic theory because, even if the general approach of kinetic theory was adopted, the resulting framework is substantially different from kinetic theory. Such a difference originates from the adoption of different forms of interactions, and only basic results of kinetic theory remain valid in the proposed framework. After a short introduction on kinetic theory of gas mixtures, we showed how to reinterpret the laws of physics in the field of opinion dynamics. In particular, we showed how to use the Boltzmann equation to study macroscopic properties of a multi-agent system and to derive the asymptotic profile of the distribution of opinion. Finally, after a brief description of various sociological phenomena involved in opinion dynamics, we showed the explicit expressions of the station-



(a) (b)

Figure 2. The stationary profiles in (35) are shown for $\lambda = 1/8$ (blue line), $\lambda = 1$ (red line), and $\lambda = 8$ (green line) and for: (a) $u = 0$; (b) $u = 1/3$.

ary profiles obtained considering two of such phenomena for multi-agent systems where all agents are equal. We showed that, if only compromise is considered, the average opinion remains constant and the variance of the opinion exponentially tends to 0 as the number of interactions increases. Hence, consensus is always reached and the opinion of each agent tends to the initial average opinion. Instead, if diffusion is considered together with compromise, stationary profiles with different characteristics can be derived depending on the parameters of the model. Similar results hold for a multi-agent system where agents are grouped into classes on the basis of the values of the parameters which characterize their propensity to change their opinions [3]. Further work on this topic involves the modeling of microscopic phenomena not yet studied using the proposed framework. As explained in the paper, the choice of different microscopic rules to update the opinions of two interacting agents has a strong impact on the collisional operator and, hence, on the macroscopic characteristics of the systems and on stationary profiles. Moreover, the inclusion of locality of interactions is a significant improvement of the proposed framework which requires adopting a non-spatially homogeneous form of the Boltzmann equation, or introducing a form of distance among agents in microscopic rules.

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