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Interpretation is an action: understanding diagrams by manipulating them

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Abstract. In this paper, I will defend the idea that *interpretation is an action*. In particular, I will argue that mathematical diagrams and figures are understood only when the user knows how to manipulate them, to the aim of making an inference and obtaining a new conclusion.

1 *SEEING AS* AND REPRESENTATIONAL SYSTEMS

The motto in the title, 'interpretation is an action', is taken from Wittgenstein. To introduce the main idea of this paper, I will present one of his numerous examples of reasoning with visual patterns [12]. I am well aware that, as Wittgenstein himself suggested, the philosopher should resist the use of a 'one-sided diet of examples'. Yet, I believe that this example alludes in a striking way at the link I am interested in between interpretation and action in diagrammatic reasoning.

Imagine two shapes. Shape A is a shape that may or may not have symmetrical properties of some kind, but most importantly it does not look familiar to the observer. That means for example that the observer cannot find a word to name it - she *does not have any name* for it. Moreover, the observer is not sure to what it alludes to, she can only guess: a lamp on a table maybe? Shape B instead is recognizable as a sequence of letters arranged in a meaningful word.

Now, imagine reversing both shapes as in Fig.1 and Fig.2[2]:

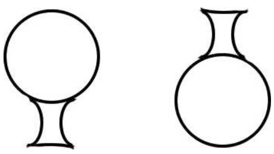

Figure 1. Shape A and its reversed version

Figure 2. Shape B and its reversed version

The difference between the visual impressions of shape B and of its reversed version is perceived as bigger than the difference between the visual impressions of shape A and of its reversed version. Why that? One reason could simply be that shape B looks more salient to the observer than shape A. Of course the observer could speculate that shape A is a picture of a lamp on a table and its reversed version a picture on how the lamp looks like if it is hanging from the ceiling. Yet, this reading would not be necessarily shared by other observers. By contrast, if we consider another salient shape such as shape C and its reversed version in Fig.3, we will see that also in this case looking at the reversed version disturbs us more than in the case of shape A.

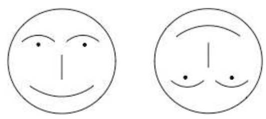

Figure 3. Shape C and its reversed version

Nevertheless, I want to suggest something more that what is happening for shape C. In fact, salience here is not enough. Or better, salience needs to be explained. My claim is that shapes are perceived as salient when the observer knows how to *use* them. To clarify, the observer who recognizes shape B or shape C is ready to *reproduce* them, in a meaningful way and without damages, far better that she could in the case their reversed version. Moreover, in the case of shape B, the observer has been *trained* to do that - that is, she has been trained to write. By contrast, in the case of shape A and its reversed version, they are both arbitrary shapes equally easy (or difficult) to be drawn[6].

The hypothesis here is that the understanding of a picture, that means its *correct interpretation*, has consequences on our *behavior*, and triggers some particular reaction in us, in what we can *do of* the picture, reproducing or manipulating it, in a meaningful way. The observer who has learnt how to write and who masters the writing system, is ready to recognize something, a word, in shape B. In the same way, shape C is meaningful, since the observer would know in which context to use it. Moreover, she would also know in which way to change it in order to change its meaning as well (as with emoticons). Shape A and its reversed version instead, do not recall any system of representation to the observer and as a consequence any kind of possible meaningful manipulation.

Can we assume this kind of approach also for *mathematical* diagrams and figures? To explore this possibility, I will introduce in the following sections the notion of *constructional* drawings and consider its relationship with mathematical practice.

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[2] Actually, Wittgenstein reverses shape A and reflects shape B as in mirror, but I will reverse both to avoid incongruencies.

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2 THE NOTION OF CONSTRUCTIONAL DRAWINGS

The hypothesis behind this paper is that what is relevant in discussing diagrammatic reasoning in mathematics is the *use* of diagrams and figures more than the individuation of their propositional content. As Avigad suggests ascriptions of understanding are best understood in terms of the possession of certain *abilities*. It is an important philosophical task to try to characterize the relevant abilities in sufficiently restricted contexts in which such ascriptions are made. This is a central issue also for automated formal verification, where, as he suggests, when it comes to verifying an inference, different sorts of agents will have different strengths and weaknesses. As humans, we spontaneously refer to diagrams because we are good at recognizing symmetries and relationships in information so represented; machines, in contrast, can keep track of gigabytes of information and carry out exhaustive computation where we are forced to rely on little tricks. If we are interested in differentiating good teaching practice from good programming practice, we have no choice but to relativize our theories of understanding to the particularities of the relevant class of agents [1].

Let us consider a useful analogy and take into account drawings for technical uses, e.g. drawings that are behind the design of an artifact. In spite of their appearance, these drawings are not depictive but serve as *inferential* or *prescriptive tools*. An effective technical drawing has the property of 'good legibility', which is a property that does not depend upon the respect of some mathematical rule, but upon the specifics of *biologically and socially characterized* visual systems of the human users, even when these are assisted by artificial processing systems[7]. These constraints long precede the regularization of representational systems by the definition of explicit rules, such as, for example, in projections. Furthermore, thanks to these constraints, drawings can be *used* and *reproduced*. Learning to use them is learning to correctly manipulate them in order to acquire new knowledge.

The same happens for other cases of diagrams. For example, the user demonstrates to know how to use a graph when, once asked to reproduce it, in doing that she gets rid of 'chartjunk', that is of the set of all visual elements that are not necessary to comprehend the information represented and that in some cases can even be distractors [8][3]. Analogously, Euler or Venn circles surely exploit some 'visual schemas' such as *shape containment* [5]; nevertheless, they can be manipulated and reproduced correctly only when the user knows which of their spatial properties are invariant in all their possible meaningful manipulations. For example, circles can be replaced by squares, provided intersections are preserved; no color can be inserted in a Euler circle, while on the contrary the location of the color should be preserved in a Venn circle.

I will define this kind of drawings that are given as *tools* to calculate and measure, as well as to plan and compare different solutions, as *constructional drawings*. They can - and to some extent, they must - be adjusted, reconsidered, prepared in advance. In the study of drawing there has been the tendency to neglect much of what today we would call *detail drawing*, which in the past was marked on the actual stone or wood in order to produce an artifact [2]. The shapes of constructional drawings may be understood as *operations* that are not visible in the final product: it is because we are able to make the drawing that we can understand the result it designs. Until the second half of the 20th century, engineering schools have illustrated the understanding of drawings by teaching *how to make them*: training included apprenticeships in which students learnt to appreciate the nature of materials and machines through laboratory experience. The knowledge was thus based on sensory observations, and masters guided the apprentices showing them what to look for [3].

[3] Examples of chartjunk include heavy or dark lines in the graph, unnecessary text or inappropriately complex fonts, pictures or icons to represent data graphs, or ornamental shading.

3 FROM CONSTRUCTIONAL DRAWINGS TO MATHEMATICAL DIAGRAMS

Diagrams and figures in mathematics as well can be defined as *constructional*, in analogy with the cases considered in the previous section.

First of all, they are subject to constraints both as two-dimensional *physical* objects and as objects that require a user to *interpret* them. Topological relations, for example, are very basic spatial relations such as proximity or enclosure that would not change in a diagram if the diagram were printed on a rubber sheet and the sheet were stretched or twisted [10]. Nevertheless, the recognition of such spatial relationships must be accompanied by interpretation, so that the diagram can be used to the aim of obtaining new conclusions within a specific theory.

To integrate these two kinds of constraints, it is possible to refer to the notion of *manipulation practice*. A diagram is interpreted *dynamically*: informal inferences take the form of transformations. In the range of all possible transformations only some of them are legitimate within the theory considered: the rules of diagrammatic representations are therefore normally externalized as procedures, and, as a consequence, what must be learned is not abstract rules, but *instructions* on how to act on the diagram and to read and interpret it correctly. To sum up, the *correct interpretation* of a diagram gets intimately connected to the *systematic actions that are performed on it* [4].

Some seminal studies in psychology have discussed *constructive* perception, i.e. the coordination of two processes: reorganizing perception and associating ideas [9]. These processes are correlated independently with a *perceptual* ability, reorganizing parts of figures, that means grouping them in such a way that they can be reinterpreted, and with a *conceptual* ability, associative fluency. The perceptual component facilitates perceptually reorganizing what one sees in the external environment, enabling detection of subtle features and relations that novices might not discern; the conceptual component enables fluency in generating new and related thoughts. To keep generating new ideas in a domain these two components must be tightly coordinated. It would be interesting in this respect to analyze the educational software actually available to teach mathematics, logic and geometry. Does this kind of constructive perception get nurtured by using them?

Expertise is thus attained by coming to differentiate and reorganize parts, wholes, perspectives, and reference frames. Perceptual expertise must then be coordinated with behavior. The implication is that expertise is not acquired straightforwardly by teaching, but rather by a coordinated act of perceiving new features and relations previously unheeded, conceiving of new and associated thoughts previously unattended to and acting on the surrounding environment in the domain.

Some work has been done in computer science as well, to the aim of understanding reasoning and problem solving activities that combine representation and inferences in both diagrammatic and symbolic modes in such a way that different subproblems are solved by the mode that is best suited for them.

One attempt is the implementation by Chandrasekaran and colleagues, based on three dimensions and on different perceptual routines [11]. The first and most basic dimension is that of seeing the world as composed of *objects*, which are discriminated from ground and perceived as shapes. The second dimension is that of *seeing as*, e.g. recognizing an object as a telephone, or a figure in a diagram as a triangle. The recognition vocabulary comes in a spectrum of domain-specificity. The third dimension is *relational* - seeing that an object is to the left of another object, taller than another, is part of another, etc.. In this implementation, the only perceptual routines that are immediately applied as a diagram is created or changed are represented by the set of emergent object routines. By contrast, all the other routines apply only in response to a problem solving need. Furthermore, even in the case of emergent objects, the number of these objects, when linked to problem solving, can grow exponentially.

Even if their motivation is not to propose a psychological theory, but rather to expand the options available to designers of AI and human-machine systems, I think that their strategy goes in the direction I discuss here about how to deal with diagrammatic reasoning.

Let us assume then that in mathematics, the most interesting diagrams are all dynamic and constructional, that means inference-promoting and correctly understood only when reproduced and correctly manipulated. In fact, diagrams and figures would invite the user to perform some operation on them, in order to solve a problem. Going back to Wittgenstein, we could claim that one of the consequences of this view would be that mathematics is a set of techniques and not a series of true and false propositions. To clarify, the point is that mathematical understanding would not concern the knowledge of a list of propositions of which we want to know the truth value, but rather of a web of interconnected facts that results in the learning and in the application of a mathematical practice. One consequence is seeing proofs in a new light: it is the very process of giving the proof that teaches the mathematician what the proof is about.

4 A POSSIBLY MORE GENERAL CONCLUSION (work in progress)

If the use of mathematical diagrams involves the coordination of several processes (and abilities), we could go even further. In fact, three cognitive systems appear to play a role here: the *conceptual* system, the *visuo-spatial* system and the *motor* system. The hypothesis is that diagrams have been invented (or discovered) with the function of facilitating and organizing our reasoning and are used dynamically. Their use would engage all of these three systems: the same manipulation of a diagram is seen by the conceptual systems as the compliance to some visual invariance, by the visual-spatial system as a transformation in time, by the motor system as a movement or a movement plan. Dynamically interpreted diagrams work at the interface between these three systems to churn out an inference that puts forward a new conclusion.

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